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Differential concomitants of the covariant tensor b_{ij} which fulfils the condition Det $b_{(ij)} \neq 0$

INTRODUCTION. Let X^n be an n-dimensional manifold. We consider a tensor field b_{ij} on X^n. We shall assume that

$$\text{Det } b_{(ij)} \neq 0, \tag{1}$$

where

$$b_{(ij)} \overset{\mathrm{df}}{=} \frac{1}{2}\,(b_{ij}+b_{ij}).$$

If the transformation of the coordinate system has the form

$$\bar{x}^i = \bar{x}^i\,(x^k); \qquad i,\ k = 1,\ 2, \ldots, n, \tag{2}$$

then we put

$$\begin{aligned} &\text{a)}\ A^i_k = \frac{\partial \bar{x}^i}{\partial x^k} \\ &\text{b)}\ B^i_k = \frac{\partial x^i}{\partial \bar{x}^k} \\ &\text{c)}\ J = \text{Det}\, \| A^i_k \| \neq 0. \end{aligned} \tag{3}$$

The partial derivative of a function U with respect to x^l will be denoted by $U_{,l}$.

We say that a geometric object Θ is a differential concomitant of order s of the tensor b_{ij}, if for every coordinate system (x^i) we have

$$\Theta = \Theta\,(b_{ij},\, b_{ij,k}, \ldots,\, b_{ij,k_1 \ldots k_s}) \tag{4}$$

(cf. [1] p. 148, also [3] p. 138).

In the present paper we consider the differential concomitants of the first and second order of the tensor b_{ij} which are purely differential geometric objects of the first class. After a change of the coordinate system (2) the components ω of these objects are transformed according to a rule

$$\bar{\omega} = F(\omega, A), \tag{5}$$

where $A = \| A^i_k \| \in GL(n)$ and the function F satisfies the following equations

$$\begin{aligned} &\text{a)}\quad F[F(\omega, A_1), A_2] = F(\omega, A_2 A_1), \\ &\text{b)}\qquad\quad F(\omega, E) = \omega. \end{aligned} \tag{6}$$

E is the unit element of the group $GL(n)$, A_1, A_2 denote here arbitrary elements of the group $GL(n)$ and A_2A_1 denotes the product of matrices A_1, A_2.

If the tensor b_{ij} is non-singular and symmetric, then all differential concomitants of order s of the tensor b_{ij} are algebraic concomitants of the tensor b_{ij}, of curvature tensor and of its covariant derivatives (cf [3] p. 138, also [2] p. 72, 73).

In § 1 we shall determine the general form of the differential concomitant of the first order of the tensor b_{ij}, in § 2 we shall find the general form of the differential concomitant of the second order of the tensor b_{ij}.

Since $\operatorname{Det} b_{(ij)} \neq 0$, we can form the Christoffel symbols with respect to $b_{(ij)}$.

§ 1. C a s e $s = 1$. We put

$$\begin{aligned} &\text{a)}\quad g_{ij} \overset{\mathrm{df}}{=} b_{(ij)} = \frac{1}{2}(b_{ij} + b_{ij}) \\ &\text{b)}\quad c_{ij} \overset{\mathrm{df}}{=} b_{[ij]} = \frac{1}{2}(b_{ij} - b_{ji}) \end{aligned} \tag{1.1}$$

and we denote by Γ^i_{jk} the Christoffel symbols with respect to $g_{ij} = b_{(ij)}$:

$$\Gamma^i_{jk} = \frac{1}{2} g^{is} (g_{sk,\, j} + g_{js,\, k} - g_{jk,\, s}). \tag{1.2}$$

The covariant derivative of a quantity U with respect to Γ^i_{jk} will be denoted by $U_{;k}$.

Let a_{ij} be a covariant tensor of the second order. If we denote by $\bar{a}_{ij}$ the components of the tensor a_{ij} in the system $(\bar{x}^i)$. The the transformation formula of these components has the following form

$$\bar{a}_{ij} = B^s_i B^t_j a_{st} \tag{1.3}$$

We prove the following theorem:

THEOREM 1. *If a purely differential geometric object of the first class is a differential concominant of the frist order of the tensor* b_{ij}, *where Det $b_{(ij)} \neq 0$, then this object is an algebraic concominant of the tensors b_{ij} and $b_{[ij];k}$.*

PROOF. If an object ω of the first class is a differential concomitant of the first order of the tensor b_{ij}, then it must satisfy the following equation:

$$\omega(b_{ij}, b_{ij,k}) = F[\omega(b_{ij}, b_{ij,k}), A], \tag{1.4}$$

where

$$b_{ij,k} = B^s_{ik} B^t_j g_{st} + B^s_i B^t_{jk} g_{st} + B^s_i B^t_j B^r_k g_{st,r} \tag{1.5}$$

and $B^s_{ij} = \dfrac{\partial^2 x^s}{\partial \bar{x}^i \partial \bar{x}^j}$. We put

$$\Omega(g_{ij}, c_{ij}, g_{ij,k}, c_{ij,k}) \overset{\mathrm{df}}{=} \omega(g_{ij} + c_{ij}, g_{ij,k} + c_i{}^j{}_{,k}) \tag{1.6}$$

Inserting (1.6) into (1.4) we obtain

$$\Omega(g_{ij}, c_{ij}, g_{ij,k}, c_{ij,k}) = F[\Omega(g_{ij}, c_{ij}, g_{ij,k}, c_{ij,k}] \tag{1.7}$$

Putting $B = \|B^i_k\| = E$ in relations (1.3) and (1.5) we get

$$\bar{g}_{ij} = g_{ij}, \quad \bar{c}_{ij} = c_{ij} \tag{1.8}$$

$$g_{ij,k} = B^s_{ik} g_{sj} + B^s_{jk} g_{is} + g_{ij,k}, \quad c_{ij,k} = B^s_{ik} c_{sj} + B^s_{jk} c_{is} + c_{ij,k}. \tag{1.9}$$

Now we shall seek such values E^s_{ik} which satisfy the system of equations

$$\begin{aligned} &\text{a)}\ B^s_{ik} g_{sj} + B^s_{jk} g_{is} = -g_{ij,k} \\ &\text{b)}\ B^s_{ik} = B^s_{ki} \end{aligned} \tag{1.10}$$

It is known (cf. [1], p. 242) that the solution of this system is

$$B^s_{ik} = -\Gamma^s_{ik}. \tag{1.11}$$

If we substitute (1.11) into (1.9), then we obtain

$$\bar{g}_{ij,k} = -\Gamma^s_{ik} g_{sj} - \Gamma^s_{jk} g_{is} + g_{ij,k} = 0 \tag{1.12}$$

and

$$\bar{c}_{ij,k} = -\Gamma^s_{ik} c_{sj} - \Gamma^s_{jk} c_{is} + c_{ij,k} = c_{ij;k} \tag{1.13}$$

Let us substitute $A = B = E$, $B^s_{jk} = -\Gamma^s_{jk}$ into equation (1.7). Thus we have by (6b), (1.8), (1.12) and (1.13)

$$\Omega(g_{ij}, c_{ij}, g_{ij,k}, c_{ij,k}) = \Omega(g_{ij}, c_{ij}, 0, c_{ij;k}).$$

g_{ij} and c_{ij} are determined by b_{ij}. Hence we have

$$\omega(b_{ij}, b_{ij,k}) = f(b_{ij}, b_{[ij];k}) \tag{1.14}$$

This completes the proof.

For a symmetric and non-singular tensor we have the following conclusions:

COROLLARY 1. *If a purely differential geometric object of the first class is a differential concomitant of the first order of a symmetric and non-singular tensor* b_{ij}, *then this object is an algebraic concomitant of the tensor* b_{ij}.

COROLLARY 2. *There do not exist differential concomitants of the first order of the symmetric and non-singular tensor* b_{ij} *which are purely differential geometric objects of the first class.*

Proof. For a symmetric and non-singular tensor b_{ij} we have $b_{[ij];k}=0$.

§ 2. Case $s=2$. We denote by R^{s}_{jkl} the curvature tensor

(2.1) $$R^{s}_{jkl}=2\,(\Gamma^{s}_{j[k,l]}+\Gamma^{t}_{j[k}\Gamma^{s}_{t|l]})$$

and we put

(2.2) $$R_{ijkl}\overset{\mathrm{df}}{=}g_{is}R^{s}_{jkl}.$$

In this case we shall prove the following theorem:

THEOREM 2. *If a purely differential geometric object of the first class is a differential concomitant of the second order of a tensor* b_{ij}, *where* $\operatorname{Det} b_{(ij)}\neq 0$, *then this object is an algebraic concomitant of the tensors* b_{ij}, $b_{[ij];k}$, $b_{[ij];k;l}$ and R_{ijkl}.

Proof. If an object ω of the first class is a differential concomitant of the second order of the tensor b_{ij}, then it must satisfy the following equation

(2.3) $$\omega\,(\bar{b}_{ij},\bar{b}_{ij,k},\bar{b}_{ij,k,l})=F\,[\omega\,(b_{ij},b_{ij,k},b_{ij,k,l}),A],$$

where

(2.4) $$\begin{aligned}\bar{b}_{ij,k,l}={}&B^{s}_{ikl}B^{t}_{j}b_{st}+B^{s}_{ik}B^{t}_{jl}b_{st}+B^{s}_{il}B^{t}_{jk}b_{st}+B^{s}_{i}B^{t}_{jkl}\,b_{st}+\\&B^{s}_{ik}B^{t}_{j}B^{r}_{l}\,b_{st,r}+B^{s}_{i}B^{t}_{jk}B^{r}_{l}\,b_{st,r}+B^{s}_{il}B^{t}_{j}B^{r}_{k}\,b_{st,r}+B^{s}_{i}B^{t}_{jl}\,b_{st,r}+\\&B^{s}_{i}B^{t}_{j}B^{r}_{kl}\,b_{st,r}+B^{s}_{i}B^{t}_{j}B^{r}_{k}B^{n}_{l}b_{st,r,n}\end{aligned}$$

and $$B^{s}_{ikl}=\frac{\partial^{3}x^{s}}{\partial\bar{x}^{i}\,\partial\bar{x}^{k}\,\partial\bar{x}^{l}}$$

Let us put

(2.5) $$\Omega(g_{ij},c_{ij},g_{ij,k};c_{ij,k},g_{ij,k,l};c_{ij,k,l})\overset{\mathrm{df}}{=}\omega\,(g_{ij}+c_{ij},g_{ij,k}+c_{ij,k},g_{ij,k,l}+\\+c_{ij,k,l}).$$

If we substitute (2.5) into (2.3), then we obtain

$$(2.6)\qquad \Omega\,(\bar g_{ij},\ \bar c_{ij},\ \bar g_{ij,k},\ \bar c_{ij,k},\ \bar g_{ij,k,l},\ \bar c_{ij,k,l}) = $$
$$= F\,[\Omega\,(g_{ij},\ c_{ij},\ g_{ij,k},\ c_{ij,k},\ g_{ij,k,l},\ c_{ij,k,l}),\ A]\,.$$

We put $B = \|B^i_k\| = E$, $B^s_{jk} = -\Gamma^s_{jk}$ into (2.4). Since

$$b_{ij,k,l} = g_{ij,k,l} + c_{ij,k,l},$$

we have the following relations

$$(2.7)\qquad \bar g_{ij,k,l} = (B^s_{jkl} - \Gamma^t_{jk}\Gamma^s_{lt} - \Gamma^t_{kl}\Gamma^s_{jt} + \Gamma^s_{jl,k})g_{ij} + (B^s_{ikl} - \Gamma^t_{ik}\Gamma^s_{lt} -$$
$$- \Gamma^t_{lk}\Gamma^s_{it} + \Gamma^s_{il,k})g_{js}$$

and

$$(2.8)\qquad \bar c_{ij,k,l} = (B^s_{jkl} - \Gamma^t_{jl}\Gamma^s_{tk} - \Gamma^t_{kl}\Gamma^s_{jt} + \Gamma^s_{jk,l})\,g_{is} + (B^s_{ikl} - \Gamma^t_{il}\Gamma^s_{tk} -$$
$$- \Gamma^t_{kl}\Gamma^s_{it} + \Gamma^s_{ik,l}) + c_{ij,k,l}.$$

We insert

$$(2.9)\qquad B^s_{jkl} = B^s_{klj} = B^s_{ljk} = B^s_{jlk} = B^s_{lkj} = B^s_{kjl} = \Gamma^t_{jk}\Gamma^s_{lt} + \Gamma^t_{lk}\Gamma^s_{jl} - \Gamma^s_{jl,k}$$

into (2.7) and (2.8). Then for all possible permutations of j, k, l expressions (2.7) and (2.8) will take on the following values:

$$(2.10)\qquad \begin{aligned} \bar g_{ij,k,l} &= 0 \\ \bar g_{ik,l,j} &= R_{ijkl} \\ \bar g_{il,j,k} &= R_{ilkj} \\ \bar g_{ij,l,k} &= 0 \\ \bar g_{il,k,j} &= R_{iljk} \\ \bar g_{ik,j,l} &= R_{ijlk} \end{aligned}$$

$$(2.11)\qquad \begin{aligned} \bar c_{ij,k,l} &= R^s_{jkl}\,c_{is} + R^s_{ikj}\,c_{sj} + c_{ij,k,l} \\ \bar c_{ik,l,j} &= R^s_{ljk}\,c_{is} + R^s_{ilj}\,c_{sk} + c_{ik,l,j} \\ \bar c_{ij,l,k} &= 0 \qquad + R^s_{ijk}\,c_{sl} + c_{il,j,k} \\ \bar c_{il,jk} &= 0 \qquad + R^s_{ilk}\,c_{sj} + c_{ij,l,k} \\ \bar c_{il,k,j} &= R^s_{lkj}\,c_{is} + R^s_{ijl}\,c_{sk} + c_{ik,j,l} \\ \bar c_{ik,j,l} &= R^s_{jkl}\,c_{is} + R^s_{ijl}\,c_{sk} + c_{ik,j,l}\,. \end{aligned}$$

We substitute $\| A^i_j \| = \| B^i_j \| = E, B^i_{jk} = -\Gamma^i_{jk}$ and expresions given by (2.9) into equation (2.6). Thus we obtain by (6b), 1.12), (1.13), (2.10) and (2.11)

$$\Omega (g_{ij}, c_{ij}, g_{ij,k}, c_{ij,k}, g_{ij,k,l}, c_{ij,k,l}) = \\ = f (g_{ij}, c_{ij} \; c_{ij;k}, c_{ij;k:l}, R_{ijkl}),$$

since the values $R_{jkl}\, c_{is}$ are algebraic concomitants of the tensors R_{ijkl}, g_{ij} and c_{ij}.

Therefore ω depends only on the tensors b_{ij}, $b_{[ij];k}$, $b_{[ij];k,l}$, R_{ijkl}:

$$\omega (b_{ij}, b_{ij,k}, b_{ij,k,l}) = h (b_{ij}, b_{[ij];k}, b_{[ij];k,l}, R_{ijkl}).$$

This completes the proof.

For a symmetric and non-singular tensor we have the following conclusion:

COROLLARY 3. *If a purely differential geometric object of the first class is a differential concomitant of the second order of a symmetric and non-singular tensor b_{ij}, then this object is an algebraic concomitant of the tensors b_{ij} and R_{ijkl}.*

Proof. For symmetric and no-singular tensor b_{ij} we have $b_{[ij];k} = b_{[ij];k;l,} = 0$.

§ 3. Let $\omega = (a_{ij}, d_{ij})$ be a pair of tensors. We assume that tensor a_{ij} is symmetric and non-singular.

We have the follofing theorems:

THEOREM 3. *If a purely differential geometric object of the first class is a differential concomitant of the first order of a pair of tensors (a_{ij}, d_{ij}), where the tensor a_{ij} is symmetric and non-singular, then this object is an algebraic concomitant of the tensors a_{ij} and $d_{ij;k}$.*

THEOREM 4. *If a purely differential geometric object of the first class is a differential concomitant of the second order of a pair of tensors (a_{ij}, d_{ij}), where the tensor a_{ij} is symmetric and non-singular, then this object is an algebraic concomitant of the tensors a_{ij}, $d_{ij,k}$, $d_{ij;k;l}$, R_{ijkl}.*

The proof of theorem 3 (theorem 4) is quite similar to the proof of theorem 1 (theorem 2).

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KOMITANTY RÓŻNICZKOWE TENSORA KOWARIANTNEGO b_{ij}, SPEŁNIAJĄCEGO WARUNEK DET $b_{(ij)} \neq 0$

Streszczenie

W pracy została wyznaczona ogólna postać komitant różniczkowych pierwszego i drugiego rzędu tensora kowariantnego b_{ij}, którego część symetryczna jest tensorem nieosobliwym, będących obiektami geometrycznymi pierwszej klasy.

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